

ω -transitive Representations of Free Groups

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Abstract

Given a linear order Ω its automorphism group $\text{Aut}(\Omega)$ forms a lattice-ordered group via pointwise order. Assuming the continuum to be a regular cardinal, we show that *pathological* and ω -*transitive* (i.e. highly transitive) representations of free groups abound within *large* permutation groups of linear orders. Consequently, under the Generalized Continuum Hypothesis it is then true that given any linear order Ω for which $|\Omega| = \text{cof}(\Omega) = \aleph_i$ ($i \in \mathbb{N}$) then any permutation group that is large in $\text{Aut}(\Omega)$ contains an ω -transitive representation of $G_{\aleph_i^+}$ (i.e. the free group of rank 2^{\aleph_i}). In particular, and working solely within ZFC, we show that any large subgroup of $\text{Aut}(\mathbb{Q})$ (resp. $\text{Aut}(\mathbb{R})$) contains an ω -transitive and pathological representation of any free group of rank $\lambda \in [\aleph_0, 2^{\aleph_0}]$ (resp. of rank 2^{\aleph_0}). Lastly, we also find a bound on the rank of free subgroups of certain restricted direct products.

Introduction

The study of automorphism groups of linear orders effectively began in 1957 with a publication by P. M. Cohn [5] where he solved a question posed by B. H. Neumann: *Can the automorphism group, $\text{Aut}(\Omega)$, of any linear order, Ω , be ordered?* An *ordering* on $\text{Aut}(\Omega)$ ([3], [10]) is meant to convey a partial order $(\text{Aut}(\Omega), <)$ so that the order is invariant under actions of $\text{Aut}(\Omega)$ onto itself. Cohn answered this question negatively and gave sufficient and necessary conditions on Ω for rendering an order on $\text{Aut}(\Omega)$ and any *large* ([8]) subgroups thereof. A comprehensive retrospective survey by Bludov, Droste and Glass can be found in [2]. Under the assumption of the Generalized Continuum Hypothesis (GCH), Glass ([9]) showed that the free l -group F_{\aleph_α} of rank \aleph_α can be represented as an o -2 transitive l -subgroup of the automorphism group of an α -set. Over a decade later McCleary ([12]) crystalized further group theoretic properties of free l -groups by extending the above result to any free l -group. This remarkable result was achieved by exploiting, as he calls it, *the best of both worlds*; a right ordering (G_κ, \leq) (i.e. a right ordering on the free group of rank κ) for which the natural action of F_κ (in the sense of Conrad [6]) is faithful and o -2 transitive.

In this paper we show that ω -transitive (i.e. *highly transitive*) representations of free groups abound within large permutation groups of linear orders. More precisely we have the following main result, where $\mathfrak{c} = 2^{\aleph_0}$, \mathfrak{c}_n is the n^{th} successor cardinal of \mathfrak{c} and G_κ represents the free group of rank κ .

Theorem 2.5. *For a linear order Ω we have*

1. *Any large $H \leq \text{Aut}(\Omega)$ contains a pathological representation of G_λ for $2 \leq \lambda \leq \mathfrak{c}$. Moreover, if $|\Omega| \leq \lambda$ then G_λ can be represented as pathological and ω -transitive within H .*
2. *(\mathfrak{c} regular) If there exists a collection of κ disjoint intervals from Ω then for any large $H \leq \text{Aut}(\Omega)$ and any $n < \omega$ for which $\lambda = \mathfrak{c}_n \leq \kappa$ (resp. $\lambda = \mathfrak{c}_\omega \leq \kappa$) there exists*

a pathological representation of G_{λ^+} (resp. G_λ) in H . Moreover, if $\lambda = |\Omega|$ then G_λ and G_{λ^+} (resp. G_λ) can be represented as pathological and ω -transitive within H .

3. (GCH) If for some $n \leq \omega$, $|\Omega| = \text{cof}(\Omega) = \aleph_n$ then any large subgroup of $\text{Aut}(\Omega)$ contains pathological ω -transitive representations of G_{\aleph_n} . Moreover, if $n \in \mathbb{N}$ then the same is true of $G_{2^{\aleph_n}}$.

Notice that replacing Ω with \mathbb{Q} (resp. \mathbb{R}) in part (1) yields that any large subgroup of $\text{Aut}(\mathbb{Q})$ (resp. $\text{Aut}(\mathbb{R})$) contains a pathological ω -transitive representation of G_λ for $\aleph_0 \leq \lambda \leq \mathfrak{c}$ (resp. $G_\mathfrak{c}$). The method employed in this paper for constructing representations of free groups is supported by two powerful results: (a) the Ping-Pong Lemma (or Table-Tennis Lemma) and (b) a construction illustrated by Peter J. Cameron ([4]) of $G_\mathfrak{c}$ within $\text{Aut}(\mathbb{R})$. The first section of this paper exploits the former in forging representations of free groups of countable rank within large permutation groups. The latter is dealt with in the second section when constructing representations of free groups of uncountable rank. The bounds found in Theorem 2.5 are due to \aleph_ω and $\bigcup_{n \in \omega} \mathfrak{c}_n$ being *singular* cardinals and they can, in a sense, be ‘attained’ as a rank for it is possible to construct nested sequences of free groups of increasing rank within large permutation groups. In general, it is simple to see that for any group G for which there exists a nested collection of free subgroups $\{F_i \subset G \mid F_i \subseteq F_j \text{ for } i \leq j\}_{i \in I}$ then $\bigcup_{i \in I} F_i$ is free and of rank $\sup\{\text{rank } F_i\}$. That said, in general the same is not true provided the representations are not nested. The symbol \bigoplus denotes the restricted direct product in the following theorem.

Theorem 2.6. *Let $G = \bigoplus_{\beta \in \kappa} H_\beta$ for which $|H_\alpha| < \sup(|H_\beta|)_{\beta \in \kappa} = \lambda$ for any $\alpha \in \kappa$. Then for any freely generated $H \leq G$ we have that $\text{rank } H < \lambda$.*

1 Background and the countable case

The symbols κ and λ will always denote infinite cardinals and $\mathfrak{c} = 2^{\aleph_0}$. For any κ , $\lambda = \kappa^{+n}$ denotes the n^{th} successor of κ and $[\kappa]^{<\omega}$ denotes the collection of all of its finite subsets. A function $f : \kappa \rightarrow \lambda$ is *cofinal* in λ provided its range ($\text{ran}(f)$) knows no bounds in λ . The *cofinality* of any λ ($\text{cof}(\lambda)$) is the smallest κ so that there exists a function that cofinally maps κ into λ . A cardinal is *regular* if it is its own cofinality and singular otherwise. Since it is consistent with ZFC that $\mathfrak{c} = \aleph_1$ and that $\mathfrak{c} = \aleph_{\omega_1}$, regularity of \mathfrak{c} is then independent of ZFC. If $x, y \subset \kappa$ so that $|x| = |y| = \kappa$ and $|x \cap y| < \kappa$ then x and y are said to be *almost disjoint* (a. d.). A *family* of a. d. sets is a collection of pairwise a. d. sets and any such family is *maximal* (m. a. d. f.) whenever it is not contained in any other a. d. family.

Standard definitions and notation regarding permutation groups (i.e. *free l -group*, *faithful representation*, *support*, etc) can be found in [2]. In order to avoid confusion, we remark that throughout this paper all k -tuples are assumed to be ordered and the symbols P and $\Omega = (\Omega, \leq)$ will represent a permutation group and a linear order, respectively. A k -transitive permutation group P on an arbitrary Ω is one for which given any pair of k -tuples $(a_1, a_2, \dots, a_k), (b_1, b_2, \dots, b_k) \in \Omega^k$ we can find an $f \in P$ for which $(f(a_1), f(a_2), \dots, f(a_k)) = (b_1, b_2, \dots, b_k)$. Moreover, if P is k -transitive for any $k \in \mathbb{N}$ then we refer to it as an ω -transitive permutation group. A 2-transitive permutation group will be called *doubly transitive*. Notice that if P is doubly transitive and Ω contains more than two points then Ω must be a dense linear order (DLO). A faithful representation (in the sense of [12]) of a group G within a permutation group will be denoted by \hat{G} . We adopt the traditional use of G_κ to denote the free group of rank κ while F_κ is the free l -group of rank κ . For a $g \in P$, $\text{supp}(g)$ will denote its support and P will be referred to as *pathological* provided it contains no element ($\neq e$) of bounded support. Recall that any $P \leq \text{Aut}(\Omega)$ is said to be closed under *piecewise patching* precisely when given any convex subset S of Ω and coterminal sequences $\{a_i \mid i \in \mathbb{Z}\}$ and $\{b_i \mid i \in \mathbb{Z}\}$ in S with $a_i < a_{i+1}$ and $b_i < b_{i+1}$ so that $\forall i \in \mathbb{Z}$ there exists $g_i \in P$ for which $g_i([a_i, a_{i+1}]) = [b_i, b_{i+1}]$ then P also contains an

element g that acts as the identity outside S and $g(x) = g_i(x)$ provided $x \in [a_i, a_{i+1}]$. Let P be a sublattice subgroup of $\text{Aut}(\Omega)$. P is said to be *closed under disjoint patching* if for all $i \in I$, $g_i \in P$ and $\text{supp}(g_i) \cap \text{supp}(g_j) = \emptyset$ (for $j \neq i$) then $\exists g \in P$ so that

$$g(x) = \begin{cases} g_i(x) & \text{if } x \in \text{supp}(g_i) \text{ and} \\ x & \text{if otherwise.} \end{cases}$$

Any l -subgroup of $\text{Aut}(\Omega)$ closed under piecewise patching and disjoint patching is said to be *large* in $\text{Aut}(\Omega)$ (or just *large*). The letter H will be used to refer to large permutation groups exclusively. We adopt the traditional use of the restriction symbol ' \upharpoonright '. That is, for $\Lambda \subseteq \Omega$ and any $g \in \text{Aut}(\Omega)$ then $g \upharpoonright \Lambda$ (if it exists) is the unique element in $\text{Aut}(\Lambda)$ that acts like g on Λ . Lastly, for any $P \subseteq \text{Aut}(\Omega)$ and $\Lambda \subseteq \Omega$ we define $\text{Aut}_P(\Lambda) := \{g \in \text{Aut}(\Lambda) \mid \exists \bar{g} \in \text{Aut}(\Omega) \text{ so that } \bar{g} \upharpoonright \Lambda = g\}$.

Our method for constructing representations of free groups is supported by two key-stones: one is Cameron's representation of G_c within $\text{Aut}(\mathbb{R})$ while the other is a consequence of the following lemma ([7] pg. 25).

Lemma 1.1 (Ping-Pong Lemma). *Let G be a group acting on a set X . Suppose that $\{A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_n\}$ is a set of pairwise disjoint subsets of X so that for $f_1, f_2, \dots, f_n \in G$ we have:*

$$B_i^c \subseteq f_i(A_i),$$

then the set generated by $\{f_1, f_2, \dots, f_n\}$ is free.

We begin by illustrating a simple representation of G_2 within $\text{Aut}(\mathbb{R})$ for it can be easily extended to any large permutation group of a linear order¹.

Example 1.2. Let $N_n = \{[a, a+1) \mid a \equiv n \pmod{4}\}$ ($n = 0, 1, 2, 3$) and define $f \in \text{Aut}(\mathbb{R})$ so that $f[a, a+1) = [a, a + \frac{13}{4})$, $[a + \frac{9}{4}, a + \frac{10}{4})$, $[a + \frac{6}{4}, a + \frac{7}{4})$ and $[a + \frac{3}{4}, a + 1)$ when $[a, a+1) \in N_0, N_1, N_2$ and N_3 respectively. Since f is periodic with a period of 4 we can understand how f behaves on \mathbb{R} by its action on the interval $[0, 4]$.

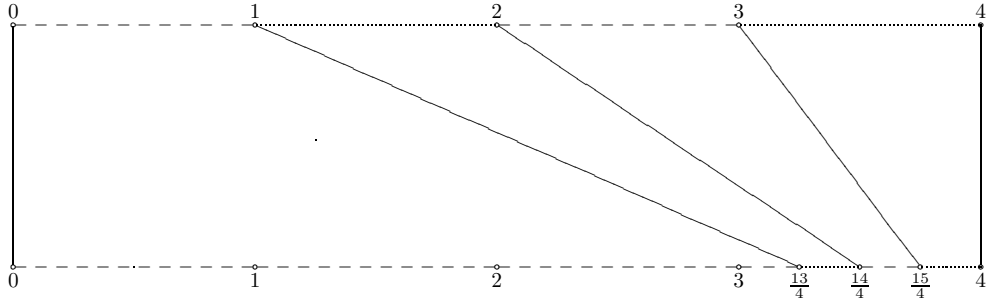


Figure 1: The interval $[0, 4]$ under f .

As for $g \in \text{Aut}(\mathbb{R})$ we let $g(x) = f(x-2)$. It is then possible to apply the Ping-Pong Lemma to this example by letting $A_1 = N_0$, $B_1 = N_3$, $A_2 = N_2$ and $B_2 = N_1$. In turn, $\{f, g\}$ generates a free subgroup of $\text{Aut}(\mathbb{R})$.

Our motives for illustrating Example 1.2 are crystalized within the proof of the following result. In order to lighten some notational burden, we use G_ω to denote G_{\aleph_0} .

¹There are several constructions of free groups within $\text{Aut}(\mathbb{R})$. Some are explicit constructions [1] while others involve more difficult algebraic notions [13].

Lemma 1.3. *For any doubly transitive $P \leq \text{Aut}(\Omega)$ closed under piecewise patching and any interval $\Lambda \subset \Omega$ there exists a representation \hat{G}_η of G_η ($1 < \eta \leq \aleph_0$) in $\text{Aut}_P(\Lambda)$ so that any element in \hat{G}_η can be trivially extended to an element in P . That is, given any $g \in \hat{G}_\eta$ we can find a $\bar{g} \in P$ so that $\bar{g} \upharpoonright \Lambda = g$ and the identity otherwise.*

Proof. Notice that it suffices to prove the above for $\eta = 2$. Let P and Λ satisfy the hypothesis, $\Gamma = \{a_i \in \Lambda \mid i \in \mathbb{Z} \text{ and } a_i < a_{i+1}\}$ (an order-isomorphic copy of \mathbb{Z} in Λ) and C_Γ denote the convex hull of Γ . In the same spirit as with Example 1.2 we let $I_n = \{[a_i, a_{i+1}] \mid a_i \in \Gamma \text{ and } i \equiv n \pmod{4}\}$. Since P is doubly transitive we can find, for all $i \equiv 0 \pmod{4}$, $f_i, f'_i \in P$ for which $[a_i, a_{i+1}] \mapsto [a_i, a_{i+3}]$ and $[a_{i-3}, a_i] \mapsto [a_{i-1}, a_i]$ respectively. In turn, if we consider the sets $\{a_i \in \Gamma \mid i \equiv 0 \text{ or } 1 \pmod{4}\}$ and $\{a_i \in \Gamma \mid i \equiv 0 \text{ or } 3 \pmod{4}\}$ and since P is closed under piecewise patching and Γ is coterminal in C_Γ then we can find an $\bar{f} \in P$ so that:

$$\bar{f}(x) = \begin{cases} x & \text{if } x \notin C_\Gamma, \\ f_i(x) & \text{if } x \in [a_i, a_{i+1}], \text{ and} \\ f'_i(x) & \text{if } x \in [a_i, a_{i-3}]. \end{cases}$$

In much the same way as with \bar{f} , let $g_i, g'_i \in P$ for which $[a_i, a_{i+1}] \mapsto [a_i, a_{i+3}]$ and $[a_{i-3}, a_i] \mapsto [a_{i-1}, a_i]$ respectively, provided $i \equiv 2 \pmod{4}$. Considering the sets $\{a_i \in \Gamma \mid i \equiv 2 \text{ or } 3 \pmod{4}\}$ and $\{a_i \in \Gamma \mid i \equiv 0 \text{ or } 3 \pmod{4}\}$ and by doubly transitivity of P we have a $\bar{g} \in P$ for which:

$$\bar{g}(x) = \begin{cases} x & \text{if } x \notin C_\Gamma, \\ g_i(x) & \text{if } x \in [a_i, a_{i+1}], \text{ and} \\ g'_i(x) & \text{if } x \in [a_i, a_{i-3}]. \end{cases}$$

By design, the restrictions of \bar{g} and \bar{f} to C_Γ exist; denote them by g and f , respectively. By applying the Ping-Pong Lemma to f and g we see that they generate a representation of G_2 within $\text{Aut}(C_\Gamma)$; let $A_1 = I_0$, $B_1 = I_3$, $A_2 = I_2$ and $B_2 = I_1$. In turn, their trivial extensions to Λ (i.e. $\bar{g} \upharpoonright \Lambda$ and $\bar{f} \upharpoonright \Lambda$) also generate a representation of G_2 within $\text{Aut}(\Lambda)$. To finish the proof we must only notice that \bar{g} and \bar{f} act as the identity outside Λ . \square

Corollary 1.4. *For Ω a linear order, any large subgroup of $\text{Aut}(\Omega)$ contains a pathological representation of G_ω .*

Proof. Let $H \leq \text{Aut}(\Omega)$ be large and $\mathcal{C} = \{\Lambda_i \subset \Omega \mid i \in I\}$ be any unbounded (below and above) collection of pairwise disjoint intervals. By Lemma 1.3 we know that for each $i \in I$ there exists a representation of G_ω in $\text{Aut}_H(\Lambda_i)$, say G_i , so that any element in G_i can be trivially extended to all of Ω ; for each $i \in I$ let $\{g_{i,j}\}_{j \in \omega}$ and $\{\bar{g}_{i,j}\}_{j \in \omega}$ denote the generating set of G_i and their trivial extensions to Ω , respectively. Since H is closed under disjoint patching then for each $j \in \omega$ we can construct an $f_j \in H$ for which $f_j(x) = \bar{g}_{i,j}(x)$ provided that $x \in \Lambda_i$ (for some $i \in I$) and the identity otherwise. This is indeed possible since the supports of any pair $\bar{g}_{k,j}, \bar{g}_{l,j}$ (with $k \neq l$) are disjoint. By assumption, \mathcal{C} is unbounded and consequently so is the support of each f_j . To this end we must only notice that $\{f_j\}_{j \in \omega}$ generates a pathological representation of G_ω in H . \square

2 The uncountable case and transitivity

As this section illustrates, Corollary 1.4 can be much improved upon. That said, the countable case is as far as the Ping-Pong Lemma can take us. For the uncountable case we turn to a construction by Cameron [4] of the free group of size continuum within $\text{Aut}(\mathbb{Q})$. We exploit such a construction by generalizing it to a certain collection of regular cardinals. The following theorem can be found in [11] and is key to what follows.

Theorem 2.1. *For $\kappa \geq \omega$ a regular cardinal*

- *If $\mathcal{A} \subset \mathcal{P}(\kappa)$ is an a. d. family and $|\mathcal{A}| = \kappa$ then \mathcal{A} is not maximal.*
- *There is a m. a. d. f. within $\mathcal{P}(\kappa)$ with cardinality κ^+ .*

In the case of \aleph_0 it is possible to construct an a. d. family within $\mathcal{P}(\aleph_0)$ of size 2^{\aleph_0} . It is this fact (along with some rather clever ideas) that enables Cameron² to construct G_c within $\text{Aut}(\mathbb{Q}, <)$. Upon careful inspection of such a construction a moment's thought should suffice to notice that the same can be done with any large subgroup of $\text{Aut}(\Omega)$, where Ω is any linear order. The reader who is familiar with the aforementioned would benefit by skipping the proof of the following theorem for it is almost a faithful copy of the original.

Lemma 2.2. *Let Ω be a linear order and $H \leq \text{Aut}(\Omega)$ be large. For any interval $\Lambda \subset \Omega$ there exists a representation \hat{G}_c of G_c in H for which any element in \hat{G}_c is the identity outside Λ .*

Proof. Begin by taking any interval $\Lambda \subseteq \Omega$ and a copy of \mathbb{N} , $\Gamma = \{a_i \in \Lambda \mid i \in \mathbb{N} \text{ and } a_i < a_{i+1}\}$, in Λ . By virtue of Lemma 1.3 and for each $i \in \mathbb{Z}$ we are guaranteed a representation of G_ω within $\text{Aut}_H(a_i, a_{i+1})$ which can be trivially extended to a representation, G_i , of G_ω in H . For each i , let $f_{i,0}, f_{i,1}, \dots$ be a set of generators of G_i . Next take any a. d. family $\mathcal{A} = \{A_\gamma \subset \mathbb{N} \mid \gamma \in \mathfrak{c}\}$ and for each $\gamma \in \mathfrak{c}$ let $h_\gamma : \mathbb{N} \rightarrow A_\gamma$ be the function that enumerates A_γ . The crucial step in this proof is to let g_α , for any $\alpha \in \mathfrak{c}$, be the permutation on Ω that when restricted to (a_i, a_{i+1}) is exactly $f_{i, h_\alpha(i)}$ and the identity otherwise. This is possible since H is closed under disjoint patching and for $k \neq l$, $\text{supp}(f_{l,n}) \cap \text{supp}(f_{k,m}) = \emptyset$. To check that indeed $\{g_\alpha \mid \alpha \in \mathfrak{c}\}$ generates G_c in H take any word $w(g_{\alpha_0}, \dots, g_{\alpha_n})$ on distinct $\alpha_0, \dots, \alpha_n \in \mathfrak{c}$. Recall that by almost disjointness of \mathcal{A} we have that $|A_{\alpha_0} \cap \dots \cap A_{\alpha_n}| \in \mathbb{N}$ and consequently any pair $g_{\alpha_k}, g_{\alpha_l}$ can agree on at most finitely many intervals. In other words, $g_{\alpha_k} \upharpoonright (a_i, a_{i+1}) = g_{\alpha_l} \upharpoonright (a_i, a_{i+1})$ for at most finitely many $i \in \mathbb{N}$. Clearly, this argument also holds for any finite collection of elements from $\{g_\alpha \mid \alpha \in \mathfrak{c}\}$ and the proof is complete. □

Just as was done with Corollary 1.4 it is then possible to extend the previous lemma to a pathological representation of G_c within any large subgroup of $\text{Aut}(\Omega)$. Indeed, we must only take an unbounded (above and below) collection of disjoint intervals from Ω and notice that for each such interval, say Λ , and any large H , Lemma 2.2 can be easily applied to $\text{Aut}_H(\Lambda)$. It is then only a matter of routine to exhibit a choice function in very much the same spirit as with Corollary 1.4.

Corollary 2.3. *For Ω a linear order we have that any large subgroup of $\text{Aut}(\Omega)$ contains a pathological representation of G_c .*

What is not obvious is that the above representation can be designed to be ω -transitive provided $|\Omega| \leq \mathfrak{c}$. Recall that any 2-transitive l -group is ω -transitive. Consequently, any large permutation group (on a linear order) is then ω -transitive. In view of Theorem 2.5, Theorem 2.4 might seem redundant. That said, we think it greatly facilitates the understanding of Theorem 2.5. The technique employed in proving the following is modeled on a result by McCleary ([12] pg. 2).

²In fact, Cameron credits such a construction to Jim Kister

Theorem 2.4. *Let $|\Omega| \leq \mathfrak{c}$ and $H \leq \text{Aut}(\Omega)$ be large. Then G_λ can be represented as a pathological ω -transitive permutation group within H provided $\mathfrak{c} \geq \lambda \geq |\Omega|$. In particular, any large subgroup of $\text{Aut}(\mathbb{Q})$ (resp. $\text{Aut}(\mathbb{R})$) contains a pathological ω -transitive representation of G_λ (resp. $G_\mathfrak{c}$).*

Proof. We prove the theorem for 2-transitivity for it can be easily extended to any ω -transitivity. Let H and λ be given, \mathcal{C} be an unbounded (above and below) collection of pairwise disjoint bounded intervals from Ω and denote $\Omega(<) = \{(a, b) \in \Omega^2 \mid a < b\}$. Let $F : \Omega(<) \times \Omega(<) \rightarrow \lambda$ be any injection. For any pair $(a_1, a_2), (b_1, b_2) \in \Omega(<)$ take any bounded interval Λ containing the points a_1, a_2, b_1 and b_2 . Since H is large, then there exists an $f \in H$ for which $f(a_i) = b_i$ and $f(x) = x$ whenever $x \in \Omega \setminus \Lambda$. For each interval $I_j \in \mathcal{C}$, let $\{f_{i,j}\}_{i \in \lambda} \subset H$ be the generating set of a representation of G_λ for which any generator is the identity outside I_j . This is indeed possible by virtue of Lemma 2.2. Next, let $\alpha = F([(a_1, a_2), (b_1, b_2)])$ and $g_\alpha \in H$ be the one for which $g_\alpha \upharpoonright \Lambda = f \upharpoonright \Lambda$, $g_\alpha \upharpoonright I_j = f_{\alpha,j} \upharpoonright I_j$ and the identity otherwise. For all other $\beta \in \Lambda$ not in the range of F we simply let $g_\beta \upharpoonright I_j = f_{\beta,j} \upharpoonright I_j$ and the identity otherwise. Finally, the set $\{g_\alpha \mid \alpha \in \lambda\}$ generates a pathological 2-transitive representation of G_λ within H . \square

We are now ready to state and prove the main theorem.

Theorem 2.5. *For Ω a linear order we have*

1. *Any large $H \leq \text{Aut}(\Omega)$ contains a pathological representation of G_λ for $2 \leq \lambda \leq \mathfrak{c}$. Moreover, if $|\Omega| \leq \lambda$ then G_λ can be represented as pathological and ω -transitive within H .*
2. *(\mathfrak{c} regular) If there exists a collection of κ disjoint intervals from Ω then for any large $H \leq \text{Aut}(\Omega)$ and any $n < \omega$ for which $\lambda = \mathfrak{c}_n \leq \kappa$ (resp. $\lambda = \mathfrak{c}_\omega \leq \kappa$) there exists a pathological representation of G_{λ^+} (resp. G_λ) in H . Moreover, if $\lambda = |\Omega|$ then G_λ and G_{λ^+} (resp. G_λ) can be represented as pathological and ω -transitive within H .*
3. *(GCH) If for some $n \leq \omega$, $|\Omega| = \text{cof}(\Omega) = \aleph_n$ then any large subgroup of $\text{Aut}(\Omega)$ contains pathological ω -transitive representations of G_{\aleph_n} . Moreover, if $n \in \mathbb{N}$ then the same is true of $G_{2^{\aleph_n}}$.*

Proof. Theorem 2.4 proves (1). For (2), we first prove the above for all $n \in \mathbb{N}$ by induction on n . Let $\mathfrak{c} = \lambda \leq \kappa$ and notice that since λ is regular then by Theorem 2.1 there exists an a. d. family $\mathcal{A} = \{A_\gamma \mid \gamma \in \lambda^+\} \subset \mathcal{P}(\lambda)$. Further, let $H \leq \text{Aut}(\Omega)$ be large, $h_\alpha : \lambda \rightarrow A_\alpha$ be the function that enumerates $A_\alpha \in \mathcal{A}$ and $C = \{I_\alpha \subset \Omega \mid \alpha \in \kappa\}$ be a collection of pairwise disjoint intervals. By virtue of Lemma 2.2, given any interval $I \in C$ there exists $\hat{G}_\mathfrak{c} \leq H$ so that any element in $\hat{G}_\mathfrak{c}$ is the identity outside I . Take any $I_\alpha \in C_\lambda = \{I_\gamma \in C \mid \gamma \leq \lambda\}$ and let $f_{\alpha 0}, f_{\alpha 1}, \dots, f_{\alpha \omega}, \dots$ be the generators of a representation of $G_\mathfrak{c}$ in H , $G(\alpha)$, for which any element in $G(\alpha)$ is the identity outside I_α . Given any $\beta \in \lambda^+$ define g_β to be the permutation on Ω that when restricted to I_α is exactly $f_{\alpha h_\beta(\alpha)}$ and the identity for any $I_\gamma \in C \setminus C_\lambda$. Again, this is possible since intervals in \mathcal{C} are disjoint and H is closed under disjoint patching. To this end we need only check that no word $w(g_{\alpha_1}, \dots, g_{\alpha_n})$ on distinct $\alpha_1, \dots, \alpha_n \in \lambda^+$ is trivial. By almost disjointness and regularity of λ we know that $|A_{\alpha_1} \cap \dots \cap A_{\alpha_n}| < \lambda$. Now, for each $\beta \in \lambda^+$ and $I_\alpha \in C_\lambda$ the choice of generator for g_β from F_α when restricted to I_α is done by h_β (i.e. $g_\beta \upharpoonright I_\alpha = f_{\alpha h_\beta(\alpha)}$). In other words, $g_\beta \upharpoonright I_\alpha$ is the $h_\beta(\alpha)^{\text{th}}$ generator of F_α . The aforementioned tells us that for a pair $g_{\alpha_i}, g_{\alpha_j}$ their restrictions to intervals in C_λ can be the same on at most fewer than λ -many intervals. As a matter of fact, they can agree on at most as many elements as A_{α_i} and A_{α_j} have in common. Clearly the argument also holds for any finite collection of elements from λ^+ . In turn, we have a $\gamma \in \lambda$ so that $w(g_{\alpha_1}, \dots, g_{\alpha_n}) \upharpoonright I_\gamma$ is not the identity.

The inductive step is handled in much the same manner. Let $\mathfrak{c} < \lambda = \mathfrak{c}_n \leq \kappa$ and notice that since successor cardinals are regular then so must be λ . Consequently, the logic behind the base case applies in spades to the inductive step.

For the case where $\lambda = \mathfrak{c}_\omega \leq \kappa$ we will construct a nested sequence of free groups within H of increasing rank. Take $\mathcal{C} = \{I_\alpha \mid \alpha \in \mathfrak{c}_\omega\}$ to be a collection of \mathfrak{c}_ω pairwise disjoint bounded intervals in Ω . For the sake of simplicity, partition \mathcal{C} into ω many parts of size \mathfrak{c}_ω so that each part is unbounded in Ω . That is, calling each part C_n ($n \in \omega$), for any $a \in \Omega$ there exist intervals $I_a, I^a \in C_n$ so that $I^a > a$ and $I_a < a$. Notice that by virtue of the previous paragraph, for any $n \in \omega$ we can concoct a representation $\hat{G}_{\mathfrak{c}_n} \leq H$ of $G_{\mathfrak{c}_n}$ so that each $g \in \hat{G}_{\mathfrak{c}_n}$ has unbounded support and acts as the identity outside $\bigcup C_n$. For each $n \in \omega$ let $\{f_{n,j}\}_{j \in \mathfrak{c}_n}$ denote a generating set of $\hat{G}_{\mathfrak{c}_n}$. Of course, each $f_{n,j}$ has unbounded support. Next, for all $j \in \mathfrak{c}$ let $g_{0,j} \in H$ be the one for which $g_{0,j}(x) = f_{n,j}(x)$ when $x \in \bigcup C_n$ and the identity otherwise. In general, we want for all $k \in \omega$

$$g_{k,j}(x) = \begin{cases} f_{n,j}(x) & \text{if } x \in \bigcup C_n \text{ and } j \in \mathfrak{c}_n \\ x & \text{otherwise.} \end{cases}$$

Consequently, for each $k \in \omega$, $\{g_{k,j}\}_{j \in \mathfrak{c}_k}$ generates a pathological copy of $G_{\mathfrak{c}_k}$ in H and for $m < n$ we have that $\{g_{m,j}\}_{j \in \mathfrak{c}_m} \subset \{g_{n,j}\}_{j \in \mathfrak{c}_n}$. To this end it should be clear that $\bigcup_{n \in \omega} \{g_{n,j}\}_{j \in \mathfrak{c}_n}$ generates a pathological copy of $G_{\mathfrak{c}_\omega}$ in H .

Next, if $|\Omega| = \lambda$ it is a simple matter to concoct a choice function between generators of G_λ and ordered pairs within Ω in much the same fashion as with Theorem 2.4.

Lastly, for (3) since G.C.H. implies that \mathfrak{c} is regular then (2) \Rightarrow (3). \square

The above construction has a serious limitation (i.e. singular cardinals) and it is unknown to the present author whether or not it can be extended any further. For some $n \in \mathbb{N}$, let $L(\aleph_n) = \aleph_n \times \aleph_n$ ordered lexicographically. Assuming G.C.H. and by Theorem 2.5 there exists a representation of the free group of rank $\aleph_n^+ = 2^{\aleph_n} = |\text{Aut}(L(\aleph_n))|$ within any large subgroup of $\text{Aut}(L(\aleph_n))$. Thus, under G.C.H., $|H| = |\text{Aut}(L(\aleph_n))|$ for any large H .

As aforesaid, nested sequences of free groups are essential for the construction of G_{\aleph_ω} in Theorem 2.5. Take for instance the following theorem where \bigoplus denotes a restricted direct product.

Theorem 2.6. *Let $G = \bigoplus_{\beta \in \kappa} H_\beta$ for which $|H_\alpha| < \sup(|H_\beta|)_{\beta \in \kappa} = \lambda$, for any $\alpha \in \kappa$. Then for any freely generated $H \leq G$ we have that $\text{rank}(H) < \lambda$.*

Proof. Assume that there exists $H \leq G$ freely generated by F_H with $|F_H| = \lambda$. In order to avoid trivialities we assume $\lambda > \aleph_0$. Let us begin by defining $i : F_H \rightarrow [\omega]^{<\omega}$ where $a \in i(g)$ iff $\pi_a(g) \neq e$ (i.e. the a^{th} coordinate of g is not the identity). Take any $A \in \text{ran}(i)$ and notice that for any $B \in \text{ran}(i)$, $A \cap B \neq \emptyset$. Lest, all elements from $i^{-1}(A)$ commute with any element from $i^{-1}(B)$. Define, for all $a \in \kappa$ and $h \in H_a$, $a(h) = \{g \in F_H \mid \pi_a(g) = h\}$ and given $B \in \text{ran}(i)$ let $B_a(h) = i^{-1}(B) \cap a(h)$. Back to A , we have that since $A \in [\kappa]^{<\omega}$ then by the Pigeonhole Principle there exists $a_1 \in A$ and $h_1 \in H_{a_1}$ for which

$$\left| \bigcup_{B \in \text{ran}(i)} B_{a_1}(h_1) \right| = \lambda$$

where we let a_1 be the smallest element of A for which the above is true. Next, denote $\mathcal{B}_1 = \bigcup_{B \in \text{ran}(i)} B_{a_1}(h_1)$ and notice that there must exist λ elements $f \in \mathcal{B}_1$ so that $i(f) \not\subset A$ (since the largest element in $\{|H_a|\}_{a \in A}$ is smaller than λ). Take any pair $g_1, g_2 \in \mathcal{B}_1$ and observe that if $i(g_1) \cap A = i(g_2) \cap A = \{a_1\}$ then $[g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$ commutes with

all elements from $i^{-1}(A)$. Thus, for all but at most one $g \in \mathcal{B}_1$, $i(g) \cap A - \{a_1\} \neq \emptyset$. Let $A_1 = A - \{a_1\}$. Since $A_1 \in [\kappa]^{<\omega}$ then we can find an $a_2 \in A_1$ and $h_2 \in H_{a_2}$ so that

$$\left| \bigcup_{B \in \text{ran}(i)} (B_{a_1}(h_1) \cap B_{a_2}(h_2)) \right| = \lambda.$$

Again, we let a_2 be the smallest element in A_1 that satisfies the above equation. We are now in a very similar situation to that encountered at the beginning of the proof. In turn, we can run the above argument until we exhaust all of A , at which point we have a collection of elements from H whose commutator commute with any element from A . Indeed, let us assume that $A = \{a_1, \dots, a_k\}$ and that

$$\left| \bigcup_{B \in \text{ran}(i)} \left(\bigcap_{j \leq k-1} B_{a_j}(h_j) \right) \right| = \lambda \text{ for } h_j \in H_{a_j}.$$

Let $\mathcal{B}_k = \bigcup_{B \in \text{ran}(i)} \left(\bigcap_{j \leq k} B_{a_j}(h_j) \right)$ and again take any pair $g_1, g_2 \in \mathcal{B}_k$ for which $i(g_1)$ and $i(g_2)$ are not contained in A . If $a_k \notin i(g_1), i(g_2)$ then $[g_1, g_2]$ commutes with everything in A . In turn, we must have that for all (but at most one) $g \in \mathcal{B}_k$, $a_k \in i(g)$. Hence, for the last element in A we have

$$\left| \bigcup_{B \in \text{ran}(i)} \left(\bigcap_{j \leq k} B_{a_j}(h_j) \right) \right| = \lambda \text{ for } h_j \in H_{a_j}.$$

Lastly, take any pair $g_1, g_2 \in \bigcup_{B \in \text{ran}(i)} \left(\bigcap_{j \leq k} B_{a_j}(h_j) \right)$ for which $i(g_1), i(g_2) \not\subseteq A$ and notice that $[f, [g_1, g_2]] = e$ for any $f \in i^{-1}(A)$. □

Corollary 2.7. *Let $G = \bigoplus_{i \in \omega} F_i$ so that each F_i is the free group of rank \aleph_i . If $H \leq G$ is a free group then $\text{rank}(H) < \aleph_\omega$.*

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